7.2 Equivalent Vector Channel

	w	
00	1	$s_1(t)$
01	2	$s_2(t)$
11	3	$s_3(t)$
10	4	$s_4(t)$

7.21. Recall that we are considering the digital modulator/demodulator part shown in Figure 24.

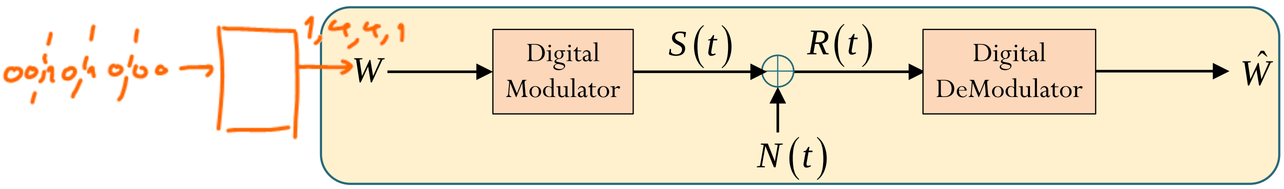


Figure 24: Digital modulator/demodulator and the waveform channel

- The input of the modulator is the (random) message (index) $W \in \{1, 2, \ldots M\}$.
  - Prior probabilities: $p_j = P[W = j]$.
- Each message is mapped to a waveform to be transmitted over the waveform channel as the transmitted waveform $S(t)$.
  - There are M possible messages. So, there are M waveforms:

    $$s_1(t), s_2(t), \ldots, s_M(t).$$

    The (symbol) energy of the j-th waveform is $E_j = \langle s_j(t), s_j(t) \rangle$.

    The average energy per symbol is $E_s = \sum_{j=1}^{M} p_j E_j$.
  - Transmission of the message $W = j$ is done by inputting the corresponding waveform $s_j(t)$ into the channel.

    Therefore, the probability that the waveform $s_j(t)$ is selected to be transmitted is the same as the probability that the j-th message occurs:

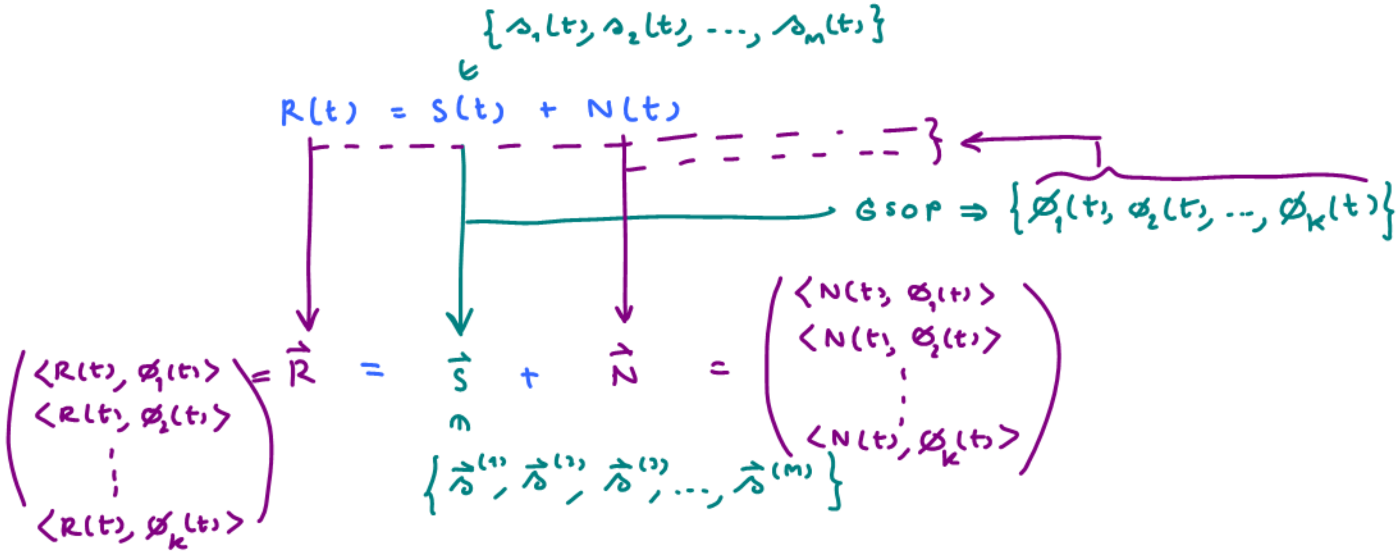
    $$p_j = P[W = j] = P[S(t) = s_j(t)]$$
- The noise $N(t)$ in the channel is assumed to be additive. So, the receiver observes $R(t) = S(t) + N(t)$. The noise is also assumed to be independent from the transmitted waveform $S(t)$.

Definition 7.22. Conversion of Waveform Channels to Vector Channels:

(a) Given M waveforms $s_1(t), s_2(t), \ldots, s_M(t)$, first find (possibly by GSOP) the K orthonormal basis functions $\phi_1(t), \phi_2(t), \ldots, \phi_K(t)$ for the space spanned by $s_1(t), s_2(t), \ldots, s_M(t)$.

(b) Next, convert the waveforms $S(t)$, $R(t)$, and $N(t)$ to their corresponding vectors $\mathbf{S}$, $\mathbf{R}$, and $\mathbf{N}$ by performing inner-product with the orthonormal basis functions: the i-th component of the vector is the inner-product between the waveform and $\phi_i(t)$. In particular,

$$S_i = \langle S(t), \phi_i(t) \rangle, \quad R_i = \langle r(t), \phi_i(t) \rangle, \quad N_i = \langle N(t), \phi_i(t) \rangle.$$



Remarks:

- We use letter K instead of letter N to represent the number of orthonormal basis functions to avoid the confusion with the random noise which is also denoted by letter N.
- This conversion is the same as what we did when we convert waveforms to vectors via the GSOP. (See Eq. (34) and Figure 17a.) When $s_j(t)$ is transmitted, the corresponding "transmitted" vector will be $\mathbf{s}^{(j)}$.

7.23. Some facts:

(a) From the perspective of designing optimal demodulator, the waveform channel and the vector channel are "equivalent".

(b) For $R(t) = S(t) + N(t)$, we have $\mathbf{R} = \mathbf{S} + \mathbf{N}$.

(c) $E_j = \langle s_j(t), s_j(t) \rangle = \langle \mathbf{s}^{(j)}, \mathbf{s}^{(j)} \rangle$. $\langle r(t), s_j(t) \rangle = \langle \vec{r}, \vec{s}^{(j)} \rangle$

(d) Prior probabilities:

$$p_j = P[W = j] = P[S(t) = s_j(t)] = P\left[\mathbf{S} = \mathbf{s}^{(j)}\right]$$

(e) $\mathbf{S} \perp\!\!\!\perp \mathbf{N}$

(f) When $N(t)$ is a white noise process, with PSD $= \frac{N_0}{2}$ across all freq. (under consideration) we have

(i) $\mathbb{E}[N_j] = 0$, and

(ii) $\mathbb{E}[N_i N_j] = \begin{cases} N_0/2, & i = j, \\ 0, & i \neq j. \end{cases}$ $\quad \vec{N} = \begin{pmatrix} N_1 \\ N_2 \\ \vdots \\ N_k \end{pmatrix}$

In other words, the noise components are uncorrelated and $\mathbb{E}\left[N_i^2\right] = \operatorname{Var} N_i = \frac{N_0}{2}$.

$\operatorname{Var} X = \mathbb{E}[X^2] - (\mathbb{E}X)^2$